Parallelogram Law

A normed space $(X, \|\cdot\|)$ is an inner product space

if and only if $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$ for all x, y.

Proof : $(\Rightarrow)$ If $(X, \|\cdot\|)$ is an inner product space,

then $\|x+y\|^2+\|x-y\|^2$

$$= (x+y,\ x+y) + (x-y,\ x-y)$$

$$= \|x\|^2+\|y\|^2+(x,y)+(y,x)+\|x\|^2+\|y\|^2-(x,y)-(y,x)$$

$$= 2\|x\|^2+2\|y\|^2 \quad \text{for all } x, y \in X.$$

$(\Leftarrow)$ If $(X, \|\cdot\|)$ satisfies Parallelogram Law,

put $(x, y) = \frac{1}{4}(\|x+y\|^2-\|x-y\|^2)+\frac{i}{4}(\|x+iy\|^2-\|x-iy\|^2)$.

We need to check $\langle\cdot,\cdot\rangle$ is an inner product.

(i) $(x, x) = \frac{1}{4}(\|2x\|^2-0)+\frac{i}{4}(\|(1+i)x\|^2-\|(1-i)x\|^2)$

$$= \|x\|^2+\frac{i}{4}(2\|x\|^2-2\|x\|^2)$$

$$= \|x\|^2 \geq 0$$

$(x, x) = 0 \quad \text{iff} \quad x = 0$

(ii) For any x, y,

$$(x, y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) + \frac{i}{4}\left(\|x+iy\|^2 - \|x-iy\|^2\right)$$

$$(y, x) = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2\right) + \frac{i}{4}\left(\|y+ix\|^2 - \|y-ix\|^2\right)$$

To show $(x, y) = \overline{(y, x)}$, it suffices to show

$$\|x+iy\|^2 - \|x-iy\|^2 = -\left(\|y+ix\|^2 - \|y-ix\|^2\right), \text{ i.e.,}$$

$$\|x+iy\|^2 + \|y+ix\|^2 = \|x-iy\|^2 + \|y-ix\|^2.$$

$$\text{L.H.S.} = \frac{1}{2}\left(\|x+iy+y+ix\|^2 + \|x+iy-y-ix\|^2\right) \quad \text{PL}$$

$$= \frac{1}{2}\left(\|(1+i)(x+y)\|^2 + \|(1-i)(x-y)\|^2\right)$$

$$= \|x+y\|^2 + \|x-y\|^2$$

$$\text{R.H.S.} = \frac{1}{2}\left(\|x-iy+y-ix\|^2 + \|x-iy-y+ix\|^2\right)$$

$$= \frac{1}{2}\left(\|(1-i)(x+y)\|^2 + \|(1+i)(x-y)\|^2\right)$$

$$= \|x+y\|^2 + \|x-y\|^2$$

(iii) Wish $(x+y, z) = (x, z) + (y, z)$.

$$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x+z-y\|^2$$

$$= 2\|y+z\|^2 + 2\|x\|^2 - \|y+z-x\|^2$$

$$= \|x+z\|^2 + \cancel{\|x\|^2} + \|y+z\|^2 + \cancel{\|y\|^2} - \cancel{\tfrac{1}{2}\|x+z-y\|^2} - \cancel{\tfrac{1}{2}\|y+z-x\|^2}$$

$$\|x+y-z\|^2 = 2\|x-z\|^2 + 2\|y\|^2 - \|x-z-y\|^2$$

$$= 2\|y-z\|^2 + 2\|x\|^2 - \|y-z-x\|^2$$

$$= \|x-z\|^2 + \cancel{\|x\|^2} + \|y-z\|^2 + \cancel{\|y\|^2} - \cancel{\tfrac{1}{2}\|x-z-y\|^2} - \cancel{\tfrac{1}{2}\|y-z-x\|^2}$$

Thus $\|x+y+z\|^2 - \|x+y-z\|^2$

$$= \|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2.$$

Similarly, you can check this for the imaginary part.

(iv) Wish $(\alpha x, y) = \alpha(x, y)$. —— $(*)$

You can check by definition $(*)$ holds for $\alpha = \pm 1, \pm i$.

By (iii) and induction, $(*)$ holds for $\alpha \in \mathbb{Z} + i\mathbb{Z}$.

For $\alpha \in \mathbb{Q}$, write $\alpha = \frac{p}{q}$.

For any x, y, write $x' = \frac{x}{q}$.

Then
$$\begin{aligned} q(\alpha x, y) &= q\left(\tfrac{p}{q}x, y\right) \\ &= q(px', y) \\ &= pq(x', y) \\ &= p(qx', y) \\ &= p(x, y) \end{aligned}$$

Thus $(\alpha x, y) = \frac{p}{q}(x, y) = \alpha(x, y)$.

Similarly, you can show $(*)$ holds for $\alpha \in \mathbb{Q}[i]$.

Since $\mathbb{Q}[i]$ is dense in $\mathbb{C}$, it suffices to show for fixed x, y,

$f: \mathbb{C}\setminus\{0\} \to \mathbb{R}$ by $f(\alpha) = \frac{1}{\alpha}(\alpha x, y)$

is continuous. It follows the triangle inequality of norm.

□

Gram-Schmidt process

Let $\{x_1, x_2, \dots\}$ be a sequence of linearly independent vectors in an inner product space V.

Put $e_1 = x_1/\|x_1\|$. Define e_n inductively by

$$e_{n+1} = \frac{x_{n+1} - \sum_{k=1}^{n} (x_{n+1}, e_k) e_k}{\|x_{n+1} - \sum_{k=1}^{n} (x_{n+1}, e_k) e_k\|}.$$ Then $\{e_1, e_2, \dots\}$ is orthonormal.

Proof: Clearly, e_n is normal for any $n \in \mathbb{N}$.

We wish to show for any $n \in \mathbb{N}$,

for any $m = 1, \dots, n$, $(e_{n+1}, e_m) = 0$, i.e.

$$\left(x_{n+1} - \sum_{k=1}^{n} (x_{n+1}, e_k) e_k, \; e_m\right) = 0 \quad \text{for all } m = 1, \dots, n$$

Proof by induction:

Suppose this is true for $1, \dots, n-1$.

Then $\left(x_{n+1} - \sum_{k=1}^{n} (x_{n+1}, e_k) e_k, e_m\right)$

$$= (x_{n+1}, e_m) - \sum_{k=1}^{n} (x_{n+1}, e_k)(e_k, e_m)$$

$$= (x_{n+1}, e_m) - (x_{n+1}, e_m)(e_m, e_m)$$

$$= 1.$$

□